ELSEVIER
Journal of Mathematical Economics 33 (2000) 101–107
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

A note on the stability of a Cournot–Nash equilibrium: the multiproduct case with adaptive expectations

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Received 6 June 1997; received in revised form 1 January 1999; accepted 4 March 1999

Abstract

Recently, Zhang and Zhang [Zhang, A., Zhang, Y. (1996). Stability of a Cournot–Nash equilibrium: the multi-product case. Journal of Mathematical Economics, 26, 441–462.] have presented sufficient and necessary conditions for the asymptotical stability of dynamic multiproduct oligopolies with Cournot expectations. First, we show via a counterexample that the necessary conditions given in that paper do not necessarily hold, and then their sufficient conditions are extended to adaptive expectations. Simple sufficient conditions are finally given by using special matrix norms. © 2000 Elsevier Science S.A. All rights reserved.

JEL classification: C62

Keywords: Cournot–Nash equilibrium; Stability; Adaptive expectations

1. Introduction

The stability of dynamic multiproduct oligopolies was recently examined by Zhang and Zhang (1996), who have presented sufficient conditions for the

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PII: S0304-4068(99)00007-5

asymptotical stability of Nash–Cournot equilibria. These results are the nonlinear extensions of the linear models given in (Okuguchi and Szidarovszky, 1990), and are also analogous to the nonlinear models given in the same book. Zhang and Zhang have considered only the case of Cournot expectations, and their necessary conditions were based on a result the proof of which has an error. In this paper the stability of dynamic multiproduct oligopolies will be investigated under adaptive expectations. The adaptive scheme of this paper is the same as the one given earlier in (Okuguchi and Szidarovszky, 1990). Our analysis will be based on the following theorem.

Proposition 1.1: *Let $\underline{z}^*$ be an interior equilibrium of the discrete dynamic system $\underline{z}_{t+1} = \underline{T}(\underline{z}_t)$, where $\underline{T}: D \mapsto D$ is continuously differentiable with $D \subseteq \mathbb{R}^N$ being an arbitrary set. Let $\underline{T}'$ denote the Jacobian of $\underline{T}$.*

1. *If all eigenvalues of $\underline{T}'(\underline{z}^*)$ are inside the unit circle, then $\underline{z}^*$ is locally asymptotically stable;*
2. *If at least one eigenvalue of $\underline{T}'(\underline{z}^*)$ is outside the unit circle, then $\underline{z}^*$ is unstable.*

Part 1 is well known from systems theory, since if all eigenvalues of $\underline{T}'(\underline{z}^*)$ are inside the unit circle, then there is a matrix norm such that $\|\underline{T}'(\underline{z}^*)\| < 1$ (see for example, Ortega and Rheinboldt, 1970). Part 2 is known from the stable and unstable manifold theorem (see for example, Katok and Hasselblatt, 1997), and recently an elementary proof for this result has been given by Li and Szidarovszky (1999). We mention that part 2 can be reformulated as a necessary stability condition as follows:

2^*. *If $\underline{z}^*$ is asymptotically stable, then for all eigenvalues λ_i of $\underline{T}'(\underline{z}^*)$, $|\lambda_i| \leq 1$.*

We also mention that part 2 cannot be stated via matrix norms, since if $\underline{z}^*$ is asymptotically stable, then it is possible that for all matrix norms, $\|\underline{T}'(\underline{z}^*)\| > 1$. Such an example is given by the system with

$$\underline{T}(\underline{z}) = \begin{pmatrix} xe^{-x^2} + ye^{-y^2} \\ ye^{-y^2} \end{pmatrix}, \tag{1.1}$$

where the unique equilibrium is $x^* = y^* = 0$ and

$$\underline{T}'(\underline{z}^*) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

An elementary analysis shows that $\underline{z}^* = \underline{0}$ is asymptotically stable, the only eigenvalue of $\underline{T}'(\underline{z}^*)$ is 1 with multiplicity 2, therefore condition 2^* is satisfied. However, it can be verified that for all matrix norms, $\|\underline{T}'(\underline{z}^*)\| > 1$ (also see the proof in [Li and Szidarovszky, 1999]) contradicting the necessary stability condition given in (Zhang and Zhang, 1996).

2. Stability under adaptive expectations

As in (Zhang and Zhang, 1996), let π_i denote the profit of firm $i (i=1, 2, \dots, n)$:

$$\pi_i(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n) = \sum_{r=1}^{m} x_{ir} p_r(\underline{x}_i + \underline{y}_i) - C_i(\underline{x}_i), \tag{2.1}$$

where m is the number of products, x_{ir} is the output of firm i of product r, $\underline{x}_i = (x_{i1}, x_{i2}, \dots, x_{im})^T$ is the output vector of firm i, $\underline{y}_i = \sum_{j \neq i} \underline{x}_j$ is the output vector of the rest of the industry, p_r is the price function of product r and C_i is the cost function of firm i. The price vector is given as $\underline{p} = (p_1, p_2, \cdots, p_m)^T$.

In the case of adaptive expectations, firm $i (i=1, 2, \dots, n)$ adjusts its expectation $\underline{y}_i^E(t)$ at time period t according to relation

$$\underline{y}_i^E(t) = \underline{D}_i \sum_{j \neq i} \underline{x}_j(t-1) + (\underline{I} - \underline{D}_i)\underline{y}_i^E(t-1), \tag{2.2}$$

where $\underline{I}$ is the $m \times m$ identity matrix and $\underline{D}_i$ is an $m \times m$ positive definite (usually diagonal) matrix. We mention that the diagonal elements of $\underline{D}_i$ are called the *speeds of adjustments*. In the special case of $\underline{D}_i = \underline{I}$, adaptive expectations reduce to Cournot (or static) expectations being considered in (Zhang and Zhang, 1996).

It is assumed that at time period t, firm i selects the best reply:

$$\underline{x}_i(t) = \operatorname{argmax}\left\{ \underline{x}_i^T \underline{p}\left(\underline{x}_i + \underline{y}_i^E(t)\right) - C_i(\underline{x}_i) \right\}. \tag{2.3}$$

If π_i is strictly concave in $\underline{x}_i$ with any fixed $\underline{y}_i$, then $\underline{x}_i(t)$ is unique. Sufficient conditions implying that π_i is strictly concave in x_i are presented in (Okuguchi and Szidarovszky, 1990).

Let $\underline{R}_i(\underline{y}_i(t))$ denote the best reply mapping, then Eq. (2.1) implies that the resulted dynamic process can be formulated as follows:

$$\underline{x}_i(t) = \underline{R}_i\left(\underline{D}_i \sum_{j \neq i} \underline{x}_j(t-1) + (\underline{I} - \underline{D}_i)\underline{y}_i^E(t-1) \right),$$

$$\underline{y}_i^E(t) = \underline{D}_i \sum_{j \neq i} \underline{x}_j(t-1) + (\underline{I} - \underline{D}_i)\underline{y}_i^E(t-1), \tag{2.4}$$

where the state variables are $\underline{x}_1(t), \ldots, \underline{x}_n(t), \underline{y}_1^E(t), \ldots, \underline{y}_n^E(t)$. Notice that all state variables are m-dimensional vectors, so the dimension of system (2.4) is $2nm$. The Jacobian of this system can be written as

$$\underline{T}' = \left(\begin{array}{cccc:cccc} \underline{0} & \underline{R}_1' \cdot \underline{D}_1 & \cdots & \underline{R}_1' \cdot \underline{D}_1 & \underline{R}_1' \cdot (\underline{I} - \underline{D}_1) & & & \\ \underline{R}_2' \cdot \underline{D}_2 & \underline{0} & \cdots & \underline{R}_2' \cdot \underline{D}_2 & & \underline{R}_2' \cdot (\underline{I} - \underline{D}_2) & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ \underline{R}_n' \cdot \underline{D}_n & \underline{R}_n' \cdot \underline{D}_n & \cdots & \underline{0} & & & & \underline{R}_n' \cdot (\underline{I} - \underline{D}_n) \\ \hdashline \underline{0} & \underline{D}_1 & \cdots & \underline{D}_1 & \underline{I} - \underline{D}_1 & & & \\ \underline{D}_2 & \underline{0} & \cdots & \underline{D}_2 & & \underline{I} - \underline{D}_2 & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ \underline{D}_n & \underline{D}_n & \cdots & \underline{0} & & & & \underline{I} - \underline{D}_n \end{array} \right)$$

Thus we have the following result.

Proposition 2.1: *Assume that the multiproduct oligopoly with n firms and m products has an interior equilibrium* $\underline{z}^* = (\underline{x}_1, \ldots, \underline{x}_n, \underline{y}_1^E, \ldots, \underline{y}_n^E)^T$, *and all best reply mappings* $\underline{R}_i$ *are single valued and continuously differentiable in a neighbourhood of* $\underline{z}^*$.

1. *If all eigenvalues of* $\underline{T}'(\underline{z}^*)$ *are inside the unit circle, then* $\underline{z}^*$ *is locally asymptotically stable*;
2. *If at least one eigenvalue of* $\underline{T}'(\underline{z}^*)$ *is outside the unit circle, then* $\underline{z}^*$ *is unstable*.

The size of this matrix is $2nm \times 2nm$, which can be significantly reduced in the following way. The eigenvalue equation of matrix $\underline{T}'$ has the form

$$\sum_{j \neq i} \underline{R}_i' \cdot \underline{D}_i \underline{u}_j + \underline{R}_i' \cdot (\underline{I} - \underline{D}_i) \underline{v}_i = \lambda \underline{u}_i,$$

$$\sum_{j \neq i} \underline{D}_i \underline{u}_j + (\underline{I} - \underline{D}_i) \underline{v}_i = \lambda \underline{v}_i, \quad (i = 1,2,\ldots,n). \tag{2.5}$$

Multiply the second equation by $\underline{R}_i'$ from the left hand side and subtract the resulted equation from the first equation to have

$$\lambda(\underline{u}_i - \underline{R}_i' \underline{v}_i) = \underline{0} \quad (i = 1,2,\ldots,n). \tag{2.6}$$

If $\lambda = 0$, then this eigenvalue is inside the unit circle; otherwise, $\underline{u}_i = \underline{R}_i' \cdot \underline{v}_i$. Substituting this relation into the second equation of Eq. (2.5), we have

$$\sum_{j \neq i} \underline{D}_i \cdot \underline{R}'_j \underline{v}_j + (\underline{I} - \underline{D}_i) \underline{v}_i = \lambda \underline{v}_i \quad (i = 1,2,\ldots,n),$$

which is the eigenvalue equation of matrix

$$\underline{T}'^{*} = \begin{pmatrix} \underline{I} - \underline{D}_1 & \underline{D}_1 \cdot \underline{R}_2' & \cdots & \underline{D}_1 \cdot \underline{R}_n' \\ \underline{D}_2 \cdot \underline{R}_1' & \underline{I} - \underline{D}_2 & \cdots & \underline{D}_2 \cdot \underline{R}_n' \\ \vdots & \vdots & \cdots & \vdots \\ \underline{D}_n \cdot \underline{R}_1' & \underline{D}_n \cdot \underline{R}_2' & \cdots & \underline{I} - \underline{D}_n \end{pmatrix}. \tag{2.7}$$

Thus, we have a modification of the previous result.

Proposition 2.2: *Assume that the conditions of Proposition (2.1) are satisfied.*
1. *If all eigenvalues of $\underline{T}'^{*}(\underline{z}^{*})$ are inside the unit circle, then $\underline{z}^{*}$ is locally asymptotically stable*;
2. *If at least one eigenvalue of $\underline{T}'^{*}(\underline{z}^{*})$ is outside the unit circle, then $\underline{z}^{*}$ is unstable*.

In order to obtain practical stability conditions, we have to compute the Jacobian of the best reply mappings. Differentiating the objective function of problem (2.3) and using the fact that the optimum is interior, we see that

$$\underline{p}\left(\underline{x}_i + \underline{y}_i^E\right) + \underline{J}_{\underline{p}}^T\left(\underline{x}_i + \underline{y}_i^E\right)\underline{x}_i - \nabla C_i\left(\underline{x}_i\right) = \underline{0}, \tag{2.8}$$

where $\underline{J}_{\underline{p}}$ is the Jacobian of $\underline{p}$ and ∇ denotes gradient as a column vector. In the best reply mapping $\underline{x}_i$ is the function of $\underline{y}_i^E$, so differentiating Eq. (2.8) with respect to $\underline{y}_i^E$ gives the following relation:

$$\left(\underline{J}_{\underline{p}} + \underline{J}_{\underline{p}}^T + \sum_{j=1}^{m} x_{ij}\underline{H}_{p_j} - \underline{H}_{C_i}\right)\underline{R}_i' = -\underline{J}_{\underline{p}} - \sum_{j=1}^{m} x_{ij}\underline{H}_{p_j}, \tag{2.9}$$

where $\underline{H}_{p_j}$ and $\underline{H}_{C_i}$ are the Hessians of p_j and C_i, respectively. Assuming that the first factor of the left hand side is invertible, we have

$$\underline{R}_i' = -\left(\underline{J}_{\underline{p}} + \underline{J}_{\underline{p}}^T + \sum_{j=1}^{m} x_{ij}\underline{H}_{p_j} - \underline{H}_{C_i}\right)^{-1}\left(\underline{J}_{\underline{p}} + \sum_{j=1}^{m} x_{ij}\underline{H}_{p_j}\right). \tag{2.10}$$

In (Okuguchi and Szidarovszky, 1990), it is assumed that $\underline{p}$ and C_i $(i = 1,2,\ldots,n)$ are twice continuously differentiable, $\underline{H}_{p_j}$ and $\underline{J}_{\underline{p}} + \underline{J}_{\underline{p}}^T$ are negative semidefinite, and $\underline{H}_{C_i}$ is positive definite. Under these conditions, the inverse matrix in Eq. (2.10) exists and is negative definite, and $\underline{R}_i'$ is continuous.

Simple sufficient stability conditions can be derived by using special matrix norms, since all eigenvalues of matrix $\underline{T}'^{*}$ are inside the unit circle, if one of its norms is less than one. Selecting the block-row norm we have the condition

$$\max_i\left\{\|\underline{I} - \underline{D}_i\| + \sum_{j \neq i}\|\underline{D}_i\| \cdot \|\underline{R}'_j\|\right\} < 1, \tag{2.11}$$

and by selecting the block-column norm the condition

$$\max_j \left\{ \|\underline{I} - \underline{D}_j\| + \sum_{i \neq j} \|\underline{D}_i\| \cdot \|\underline{R}'_j\| \right\} < 1 \tag{2.12}$$

is obtained. Notice that inequality (2.11) holds, if for all i,

$$\sum_{j \neq i} \| R'_j \| < \frac{1 - \|\underline{I} - \underline{D}_i\|}{\|\underline{D}_i\|}, \tag{2.13}$$

which is satisfied if $\underline{J}_p$ is symmetric,$\underline{H}_{p_i}$ and $\underline{H}_{C_i}$ are sufficiently small, furthermore $\underline{D}_i$ is diagonal and the ratio of the smallest and largest diagonal elements of $\underline{D}_i$ is not very small. In the case of Cournot expectations $\underline{D}_i = \underline{I}$, therefore, this last condition obviously holds.

Similarly, inequality (2.12) holds, if for all j,

$$\| R'_j \| < \frac{1 - \|\underline{I} - \underline{D}_j\|}{\sum_{i \neq j} \|\underline{D}_i\|}, \tag{2.14}$$

which is satisfied under very similar conditions as in the case of relation (2.11).

The size of matrix $\underline{T}'^*$ is nm, still large. Finally, an additional reduction in the size of the eigenvalue problem will be presented. However the resulting eigenvalue problem becomes nonlinear. The eigenvalue problem of matrix $\underline{T}'^*$ can be written as

$$(\underline{I} - \underline{D}_i)\underline{u}_i + \sum_{j \neq i} \underline{D}_i \underline{R}'_j \underline{u}_j = \lambda \underline{u}_i$$

for all i. Observe that this equation can be rewritten in the following way:

$$\sum_{j=1}^{n} \underline{R}'_j \underline{u}_j = \left(\underline{I} + \underline{R}'_i + (\lambda - 1)\underline{D}_i^{-1} \right) \underline{u}_i. \tag{2.15}$$

Since the left hand side does not depend on i, we have

$$\left(\underline{I} + \underline{R}'_i + (\lambda - 1)\underline{D}_i^{-1} \right) \underline{u}_i = \left(\underline{I} + \underline{R}'_1 + (\lambda - 1)\underline{D}_1^{-1} \right) \underline{u}_1$$

for all i. Assuming that $\underline{I} + \underline{R}_i' + (\lambda - 1)\underline{D}_i^{-1}$ is invertible,

$$\underline{u}_i = \left(\underline{I} + \underline{R}'_i + (\lambda - 1)\underline{D}_i^{-1} \right)^{-1} \left(\underline{I} + \underline{R}'_1 + (\lambda - 1)\underline{D}_1^{-1} \right) \underline{u}_1.$$

Substituting this relation into Eq. (2.15), we obtain a nonlinear eigenvalue problem

$$\begin{aligned} &\Bigg\{ \left((1-\lambda)\underline{D}_1^{-1} - \underline{I} \right) + \sum_{j \neq 1} \underline{R}'_j \left(\underline{I} + \underline{R}'_j + (\lambda - 1)\underline{D}_j^{-1} \right)^{-1} \Big(\underline{I} + \underline{R}'_1 \\ &\quad + (\lambda - 1)\underline{D}_1^{-1} \Big) \Bigg\} \underline{u}_1 = \underline{0}, \end{aligned} \tag{2.16}$$

where the size of the problem is only m.

As an example, consider finally the symmetric case, when $\underline{R}'_1 = \cdots = \underline{R}'_n = \underline{R}'$ at the equilibrium and $\underline{D}_1 = \cdots = \underline{D}_n = \underline{D}$. In this case, problem (2.16) has this simple form

$$\left\{\left((1-\lambda)\underline{D}^{-1} - \underline{I}\right) + (n-1)\underline{R}\right\}\underline{u}_1 = \underline{0},$$

which is the m-dimensional eigenvalue problem of matrix $\underline{I} - \underline{D} + (n-1)\underline{DR}$.

3. Concluding remarks

Dynamic multiproduct oligopolies were examined under adaptive expectations. It was assumed that each firm forms adaptive expectations on the output of the rest of the industry. Expectations on the individual outputs of the rivals can be analysed in an analogous way to the one presented for the linear case in (Okuguchi and Szidarovszky, 1990). Sufficient and necessary conditions were derived for the (local) asymptotical stability of the equilibrium. The conditions involved the Jacobians and Hessians of the price functions and the Hessians of the cost functions at the equilibrium. If the same conditions are valid at all feasible strategies (not only at the equilibrium), then global asymptotical stability of the equilibrium is assured.

Our results are common generalizations of the similar conditions given for Cournot expectations by Zhang and Zhang (1996) as well as of those obtained for adaptive expectations in the linear case by Okuguchi and Szidarovszky (1990).

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